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DEGREE OF A HOLOMORPHIC MAP BETWEEN UNIT BALLS FROM $\mathbb{C}^2$ TO $\mathbb{C}^n$

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ABSTRACT. Let f be a rational proper holomorphic map between the unit ball in $\mathbb{C}^2$ and the unit ball in $\mathbb{C}^n$. Write

$$f = \frac{(p_1, \ldots, p_n)}{q},$$

where $p_j,\ j = 1, \ldots, n$, and q are holomorphic polynomials, with $(p_1, \ldots, p_n, q) = 1$. Recall that the degree of f is defined by

$$\deg f = \max\{\deg(p_j)_{j=1,\ldots,n}, \deg q\}.$$

In this paper, we give a bound estimate for the degree of f, improving the bound given by Forstnerič (1989).

1. Introduction

Let $B_n = \{z \in \mathbb{C}^n | \ |z| < 1\}$ be the unit ball in $\mathbb{C}^n$. It is well known ([Ru80]) that the automorphisms of B_n (that is, the set of biholomorphic maps from B_n to B_n) are linear fractional, generated by the unitary linear maps and the maps φ_a given by

$$\varphi_a(z) = \frac{a - Pz - \sqrt{(1-|a|^2)}Qz}{1 - \langle z, a\rangle}, \tag{1.1}$$

where $a \in B_n$, $\ Pz = \dfrac{\langle z, a\rangle}{|a|^2}a$, $\ Qz = z - Pz$. (Here $\langle\ ,\ \rangle$ denotes the usual scalar product in $\mathbb{C}^n$.)

In [A74] and [A77], Alexander proved that any proper holomorphic map from B_n to $B_n,\ n > 1$, is an automorphism, and thus is linear fractional.

In this paper, we are concerned with the set of *rational* proper holomorphic maps from B_N to B_M, and more precisely with their degree. In [We79], Webster showed that if $f\colon B_n \to B_{n+1},\ n \geq 3$, is a rational proper holomorphic map, then f is linear fractional. We recall the following definition.

Definition 1.1. Let $f, g\colon B_N \to B_M$ be two holomorphic maps. We say that f and g are *equivalent* if there exist σ, an automorphism of B_N, and τ, an automorphism of B_M, such that $f = \tau \circ g \circ \sigma$.

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In [Fa82], Faran showed that if $f\colon B_2 \to B_3$ is a rational proper holomorphic map, then f is equivalent to one of the following maps.

(i) $(z, w) \to (z^3, w^3, \sqrt{3}zw)$;
(ii) $(z, w) \to (z, zw, w^2)$;
(iii) $(z, w) \to (z^2, \sqrt{2}zw, w^2)$;
(iiii) $(z, w) \to (z, w, 0)$.

Furthermore, in [Fa86], Faran showed that if $f\colon B_n \to B_N,\quad N < 2n-1$, is a rational proper holomorphic map, then f is linear fractional. See also [Hu99], [Hu03]. Moreover, Huang-Ji in [HJ01] showed that if $f\colon B_n \to B_{2n-1},\quad n \geq 3$, is a rational proper holomorphic map, then f is equivalent to one of the following maps.

(i) $L(z_1, \dots, z_n) = (z_1, \dots, z_n, 0, \dots, 0)$;
(ii) $W(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, z_n z_1, z_n z_2, \dots, z_n z_n)$.

Let F be a rational holomorphic map from $\mathbb{C}^N$ into $\mathbb{C}^M$. We write

$$F = \frac{(P_1, \dots, P_M)}{R}, \tag{1.2}$$

where $P_j, j = 1, \dots, M$, and R are holomorphic polynomials and $(P_1, \dots, P_M, R) = 1$. Following Forstnerič [Fo89] and Huang-Ji [HJ01], we give the following definition.

Definition 1.2. Let $F = \dfrac{(P_1, \dots, P_M)}{R}$ be as above. The degree of F, called deg F, is the number

$$\deg F = \max\{\deg(P_j)_{j=1,\dots,M}, \deg R\}. \tag{1.3}$$

Remark 1.3. It is easy to show (Lemma 2.1) that if $f, g\colon B_N \to B_M$ are equivalent (in the sense of Definition 1.1) and rational, then deg f = deg g.

Suppose now that $f\colon B_N \to B_M$ is a rational proper holomorphic map. In [Fo89] Forstnerič gave a rough bound estimate on the degree of such a map f. More precisely, he showed that if $N = 2$, then the degree of f is bounded by $M^2(M-1)$.

We address here the following question.

($*$) *Let $f\colon B_2 \to B_n$ be a rational proper holomorphic map. What is the maximum degree possible as a function of n?*

This problem is motivated by a question of X. Huang [Hu01] regarding a conjecture of D'Angelo [DA93], stating that the maximum degree of such a map is $2n-3$. Our main result is the following.

Theorem 1.4. *Let $f\colon B_2 \to B_n$ be a rational proper holomorphic map between B_2 and B_n. Then*

$$\deg\, f \leq \frac{n(n-1)}{2}. \tag{1.4}$$

Note that, using Remark 1.3, the theorem follows from [Fa82] and [CS90] in the case where $n = 3$. (The case $n = 2$ follows from [A74] and [A77].)

The paper is organized as follows. In section 2, we introduce and explain the necessary notation and terminology needed throughout the paper. In section 3, we discuss invariants associated to holomorphic maps. Theorem 1.4 will be proved in section 4. I would like to thank Dmitri Zaitsev for many helpful discussions and comments on this paper. I am also grateful to the referee for several useful remarks.

2. Preliminaries

In this section we recall a lemma of [HJ01] which, roughly speaking, says that if we have a *uniform* bound estimate of the degree of a rational holomorphic map *on the Segre variety at any point of the Lewy hypersurface in* $\mathbb{C}^N$, then we have the same bound estimate of the degree of f on $\mathbb{C}^N$.

We have the following lemma whose proof is left to the reader.

Lemma 2.1. *Let F and G be rational holomorphic mappings from $\mathbb{C}^N$ to $\mathbb{C}^M$ such that $F = \tau \circ G \circ \sigma$, with σ and τ linear fractional. Then* deg $F =$ deg G.

Let $f\colon B_2 \to B_n$ be a rational proper holomorphic map between the unit ball in $\mathbb{C}^2$ and the unit ball in $\mathbb{C}^n$. By a result of Cima-Suffridge [CS90], f extends holomorphically up to the boundary. We still write f for the induced holomorphic nonconstant map from S_2 to S_n, where S_2 (resp. S_n) is the unit sphere in $\mathbb{C}^2$ (resp. in $\mathbb{C}^n$).

Let H_N be the Siegel upper-half space

$$H_N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N \mid \operatorname{Im} z_N > \sum_{j=1}^{N-1} |z_j|^2\}. \tag{2.1}$$

It is well known [Ru80] that the holomorphic rational mapping of degree one, $F = (F_1, \ldots, F_N)$, from $\mathbb{C}^N$ to $\mathbb{C}^N$ given by

$$F_j(z) = \frac{iz_j}{1 - z_N},\ j = 1, \ldots, N-1, \quad F_N(z) = \frac{i(z_N + 1)}{1 - z_N}, \tag{2.2}$$

where $z = (z_1, \ldots, z_N)$, maps B_N biholomorphically onto H_N, and takes the unit sphere minus the point $(0, 0, \ldots, 1)$ bijectively to the Lewy hypersurface, denoted by ∂H_N, given by

$$\operatorname{Im} z_N = \sum_{j=1}^{N-1} |z_j|^2. \tag{2.3}$$

Let $p = (p_1, \ldots, p_N) = (\tilde{p}, p_N) \in \partial H_N$. Let σ_p be the rational biholomorphic map of H_N onto H_N given by

$$\sigma_p(z_1, \ldots, z_N) = \sigma_p(\tilde{z}, z_N) = (\tilde{z} + \tilde{p}, z_N + p_N + 2i\langle \tilde{z}, \tilde{p}\rangle), \tag{2.4}$$

mapping ∂H_N into ∂H_N, and sending 0 to p.

Remark 2.2. Using Lemma 2.1, (2.1), (2.2), (2.3) and (2.4), we see that $f\colon B_2 \to B_n$ *induces* a (germ at 0 of a) nonconstant *rational* holomorphic map *of same degree,* called h, defined on $U \cap \overline{H_2}$, where U is an open neighborhood of 0 in $\mathbb{C}^2$, mapping $U \cap H_2$ into H_n, $U \cap \partial H_2$ into ∂H_n, with $h(0) = 0$. We write

$$h : \partial H_2 \longrightarrow \partial H_n. \tag{2.5}$$

For $q = (\tilde{q}, q_N)$ in $\mathbb{C}^N$, we recall that the family of Segre varieties of ∂H_N is the family of complex submanifolds given by

$$Q_q = \{(\zeta, \tau) \in \mathbb{C}^{N-1} \times \mathbb{C} | \tau = \overline{q_N} + 2i\langle \zeta, \tilde{q}\rangle\}. \tag{2.6}$$

Note that if $q = (q_1, q_2) \in \mathbb{C}^2$, the family of Segre varieties is given by

$$Q_q = \{(\zeta, \tau) \in \mathbb{C}^2 | \tau = \overline{q_2} + 2i\zeta\, \overline{q_1}\}. \tag{2.7}$$

Definition 2.3. Let F be a rational holomorphic map from $\mathbb{C}^N$ to $\mathbb{C}^M$, and let $q=(\tilde{q},q_N)$ in $\mathbb{C}^N$. The degree of $F|_{Q_q}$ is the degree of the rational holomorphic mapping in $\zeta\in\mathbb{C}^{N-1}$

$$F|_{Q_q}(\zeta)=F(\zeta,\overline{q_N}+2i\langle\zeta,\tilde{q}\rangle). \tag{2.8}$$

We recall the following lemma which is due to Huang-Ji [HJ01].

Lemma 2.4 ([HJ01])**.** *Let* $F=\dfrac{(P_1,\dots,P_M)}{R}$ *be a rational holomorphic map from* $\mathbb{C}^N$ *to* $\mathbb{C}^M$*, where* P_j*,* R *are holomorphic polynomials with* $(P_1,\dots,P_M,R)=1$ $(M>N>1)$*. Assume that for each* $q\in\partial H_N$ *close to the origin,* $\deg(F|_{Q_q})\le k$*, with* $k>0$ *a fixed integer. Then*

$$\deg\, F\le k.$$

3. Invariants associated to holomorphic maps

Let $M\subset\mathbb{C}^N$ and $M'\subset\mathbb{C}^{N'}$ be real-analytic hypersurfaces through p and p' respectively. Let $\rho(z,\bar{z})$ (respectively $\rho'(z',\bar{z'})$) be a local real-analytic defining function for M near p (respectively for M' near p'). We recall some known facts and notations. For sufficiently small neighborhoods $U\subset\mathbb{C}^N$ (respectively $U'\subset\mathbb{C}^{N'}$) of p (respectively p'), the family of Segre varieties of M at $w\in U$ (respectively M' at $w'\in U'$) is the family of complex submanifolds given by

$$Q_w=\{z\in U\mid \rho(z,\bar{w})=0\}, \tag{3.1}$$

and

$$Q'_{w'}=\{z'\in U'\mid \rho'(z',\bar{w'})=0\}. \tag{3.2}$$

Let $F\colon\Omega\subset\mathbb{C}^N\to\Omega'\subset\mathbb{C}^{N'}$ be a holomorphic map with $M\subset\Omega$, $F(M)\subset M'$ and $F(p)=p'$. For $w\in M\cap U\cap\Omega$ and $w'\in U'\cap\Omega'$, we set, following Zaitsev [Z99],

$$r^2F(Q_w):=\bigcap\{Q'_{w'}\mid F(\Omega\cap Q_w)\subset Q'_{w'}\}. \tag{3.3}$$

Definition 3.1. Let $F\colon M\to M'$ as above. The number given by

$$\nu(F):=\ \max\nolimits_{w\in M}\dim_{\mathbb{C}} r^2F(Q_w) \tag{3.4}$$

is called the degeneracy of F on M.

We have the following proposition whose proof is left to the reader.

Proposition 3.2. *The number* $\nu(F)$ *given by* (3.4) *has the following properties.*

(1) $\nu(F)$ *is independent of the choice of holomorphic coordinates.*
(2) $\nu(F)\le N'-1$.

We now return to the map given by (2.5), $h\colon\partial H_2\to\partial H_n$. For $q=(q_1,q_2)\in\mathbb{C}^2$, sufficiently close to 0, we define (in the sense of germs at 0)

$$v_k(\bar{q},\zeta):=\frac{\partial^k}{\partial\zeta^k}h(\zeta,\,\overline{q_2}+2i\zeta\,\overline{q_1}),\ \ k\in\mathbb{N}. \tag{3.5}$$

We have the following lemma.

Lemma 3.3. *Let* $q\in\mathbb{C}^2$ *be fixed, sufficiently close to* 0*. Let* $v_k(\bar{q},\zeta)$ *be given by* (3.5) *and let* $u>0$ *be a fixed integer. Assume that* $\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},\zeta),\dots,v_u(\bar{q},\zeta)\}=l$*. Then*

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},\zeta),v_2(\bar{q},\zeta),\dots,v_l(\bar{q},\zeta)\}=l. \tag{3.6}$$

Proof. We proceed by induction on

$$l = \mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},\zeta),\dots,v_u(\bar{q},\zeta)\}.$$

For $l = 1$, if $v_1(\bar{q},\zeta) \equiv 0$, then by definition $v_j(\bar{q},\zeta) \equiv 0$, for every $j = 1,\dots u$, which is a contradiction.

Assume that the lemma holds for $\tilde{l} < l$, and that, by contradiction,

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},\zeta), v_2(\bar{q},\zeta),\dots,v_l(\bar{q},\zeta)\} = \mu < l.$$

By the induction hypothesis,

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},\zeta), v_2(\bar{q},\zeta),\dots,v_\mu(\bar{q},\zeta)\} = \mu.$$

This implies that there exist ζ_0 near 0 and an open neighborhood V of ζ_0, such that we can write

$$v_{\mu+1}(\bar{q},\zeta) = \sum_{j=1}^{\mu} a_j(\zeta) v_j(\bar{q},\zeta),\ \zeta \in V, \tag{3.7}$$

with $a_j(\zeta), j = 1,\dots,\mu$, smooth. Differentiating (3.7) $u-(\mu+1)$ times, we obtain that

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},\zeta),\dots,v_u(\bar{q},\zeta)\} = \mu,$$

which is a contradiction. This completes the proof of the lemma. □

Remark 3.4. Let $q = (q_1, q_2) \in \partial H_2$. We would like to mention that h is $(n-1)$-*nondegenerate at* q, in the sense of Lamel [La01], if and only if

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},q_1), v_2(\bar{q},q_1),\dots,v_{n-1}(\bar{q},q_1)\} = n-1.$$

Proposition 3.5. *Let $h\colon \partial H_2 \to \partial H_n$ as above. Then the following are equivalent.*

(1) *There exists $q \in \partial H_2$ such that h is $(n-1)$-nondegenerate at the point q.*
(2) $\nu(h) = n-1$.

Proof. We first show that (1) implies (2). Let $q = (q_1, q_2) \in \partial H_2$ satisfying (1). Consider (in the sense of germs at 0)

$$\{q' \in \mathbb{C}^n \mid h(Q_q) \subset {Q'}_{q'}\}, \tag{3.8}$$

where ${Q'}_{q'}$ is the Segre variety of ∂H_n at q'.

Using (2.6), (2.7), and putting $h = (\tilde{h}, h_n)$, (3.8) yields

$$h_n(\zeta, \overline{q_2} + 2i\zeta\,\overline{q_1}) = \overline{{q'}_n} + 2i\langle \tilde{h}(\zeta, \overline{q_2} + 2i\zeta\,\overline{q_1}), \tilde{q}'\rangle. \tag{3.9}$$

Differentiating (3.9) k times, $k \geq 1$, with respect to ζ, we obtain that $v_k(\bar{q},\zeta)$ belongs to the hyperplane in $\mathbb{C}^n$ given by

$$\tau' = 2i\langle \zeta', \tilde{q}'\rangle, \tag{3.10}$$

and therefore, by assumption, we obtain $\nu(h) = n-1$. This completes the proof that (1) implies (2).

To show that (2) implies (1), we assume by contradiction that, for every $q \in \partial H_2$,

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},q_1),\dots,v_{n-1}(\bar{q},q_1)\} < n-1.$$

Let $q_0 \in \partial H_2$ such that

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q}_0,q_{01}),\dots,v_{n-1}(\bar{q}_0,q_{01})\} = \max_{q\in\partial H_2}\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q},q_1),\dots,v_{n-1}(\bar{q},q_1)\}.$$

Let

$$l := \mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q}_0,q_{01}),\dots,v_{n-1}(\bar{q}_0,q_{01})\}.$$

The uniqueness theorem implies that

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q}_0,\zeta),\ldots,v_{n-1}(\bar{q}_0,\zeta)\} = l. \tag{3.11}$$

By Lemma 3.3, we obtain

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q}_0,\zeta),v_2(\bar{q}_0,\zeta),\ldots,v_l(\bar{q}_0,\zeta)\} = l. \tag{3.12}$$

Using (3.11), (3.12) and the uniqueness theorem, we obtain that there exists a nonempty open set U of ∂H_2 such that, for every $p \in U$,

$$\begin{aligned} &\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{p},p_1),v_2(\bar{p},p_1),\ldots,v_l(\bar{p},p_1)\} \\ &\quad = \mathrm{rk}_{\mathbb{C}}\{v_1(\bar{p},p_1),v_2(\bar{p},p_1),\ldots,v_{n-1}(\bar{p},p_1)\} = l. \end{aligned} \tag{3.13}$$

Using (3.13), we obtain that for $j \in \mathbb{N}$,

$$v_j(\bar{p},\zeta) \in \mathrm{span}_{\mathbb{C}}\{v_1(\bar{p},\zeta),v_2(\bar{p},\zeta),\ldots,v_l(\bar{p},\zeta)\}. \tag{3.14}$$

Using (3.5), (3.14) and the Taylor expansion of $f(Q_p)$ at $h(p_1,\overline{p_2}+2ip_1\overline{p_1})$, we obtain

$$h(Q_p) \subset \mathrm{span}_{\mathbb{C}}\{v_1(\bar{p},p_1),v_2(\bar{p},p_1),\ldots,v_l(\bar{p},p_1)\}. \tag{3.15}$$

Since the Segre varieties of ∂H_n are hyperplanes in $\mathbb{C}^n$ and $h(Q_p) \subset {Q'}_{h(p)}$, we conclude, using (3.15), that there exists q' close to $h(p)$, $q' \neq h(p)$, such that $f(Q_p) \subset {Q'}_{q'}$. This contradicts (2). The proof is then complete. □

4. Proof of Theorem 1.4

In this section we prove the following theorem whose proof will be based on Lemma 2.1 and Lemma 2.4.

Theorem 4.1. *Let $f\colon B_2 \to B_n$ be a rational proper holomorphic map between B_2 and B_n. If $\nu(f) \le n-k,\ 1 \le k \le n-1$, then*

$$\deg f \le \frac{(n-k+1)(n-k)}{2}. \tag{4.1}$$

Proof. Let $h : \partial H_2 \longrightarrow \partial H_n$ be the nonconstant rational holomorphic map induced by f, of same degree. (See Remark 2.2.)

We first assume that $\nu(h) = n-1$. By Proposition 3.5, there exists $q_0 \in \partial H_2$ such that

$$\mathrm{rk}_{\mathbb{C}}\{v_1(\overline{q_0},q_{01}),v_2(\overline{q_0},q_{01}),\ldots,v_{n-1}(\overline{q_0},q_{01})\} = n-1. \tag{4.2}$$

Using (2.4) and [La01], we may assume that $q_0 = 0$ and $h(0) = 0$. We claim that

$$\deg(h|_{Q_0}) \le \frac{n(n-1)}{2}. \tag{4.3}$$

Assuming the claim, we get, using Lemma 2.1, (2.4) and Proposition 3.2, that

$$\deg(h|_{Q_q}) \le \frac{n(n-1)}{2} \tag{4.4}$$

for every q close to 0. Using Lemma 2.4 and Lemma 2.1 we obtain (4.1). It remains then to prove the claim. Let $q \in Q_0$. Since $Q_0 = \{(z,w) \in \mathbb{C}^2 \mid w = 0\}$, we have

$$Q_q = \{(\zeta,\tau) \in \mathbb{C}^2 \mid \tau = 2i\zeta\bar{z}\}. \tag{4.5}$$

We consider (in the sense of germs at 0)

$$A \;=\; \{q' \in \mathbb{C}^N \mid h(Q_q) \subset {Q'}_{q'}\ \}. \tag{4.6}$$

Note that $h(q) \in A$. Since $h(0) = 0$, we have

$$A \subset Q'_0. \tag{4.7}$$

Note that for $q' = (\tilde{z}', 0) \in Q'_0$,

$$Q'_{q'} = \{(\zeta', \tau') \in \mathbb{C}^n \mid \tau' = 2i\langle \zeta', \tilde{z}'\rangle\}. \tag{4.8}$$

Writing $h = (\tilde{h}, h_n)$, using (4.5) and (4.8), we get that $q' \in A$ if and only if

$$h_n(\zeta, 2i\zeta\,\overline{z}) = 2i\langle \tilde{h}(\zeta, 2i\zeta\,\overline{z}), \tilde{z}'\rangle. \tag{4.9}$$

Differentiating (4.9) k times, $k \geq 1$, with respect to ζ, and putting $\zeta = 0$, we obtain that $v_k(\bar{q}, 0)$ belongs to the hyperplane in $\mathbb{C}^n$ given by

$$\tau' = 2i\langle \zeta', \tilde{z}'\rangle. \tag{4.10}$$

We define

$$v_k(\bar{z}) := v_k(\bar{q}, 0). \tag{4.11}$$

Note that $v_k(\bar{z})$, $k = 1, \ldots, n-1$, are polynomials of degree k in $\bar{z}$. Therefore, using (4.2), the fact that $h(q) \in A$, (4.10), and Cramer's rule, we obtain that $h_j(z, 0), j = 1, \ldots, n$, is a quotient of two polynomials of degree at most $1 + \cdots + (n-1) = \dfrac{n(n-1)}{2}$. This proves the claim.

Assume now that $\nu(f) = n - k,\ 1 \leq k < n - 1$. By repeating the arguments given in the proof of Proposition 3.5, we can show that there exists $q_0 \in \partial H_2$ such that

$$\begin{aligned} &\mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q}_0, q_{01}), v_2(\bar{q}_0, q_{01}), \ldots, v_{n-k}(\bar{q}_0, q_{01})\} \\ &\quad = \mathrm{rk}_{\mathbb{C}}\{v_1(\bar{q}_0, q_{01}), v_2(\bar{q}_0, q_{01}), \ldots, v_{n-1}(\bar{q}_0, q_{01})\} = n - k. \end{aligned} \tag{4.12}$$

Without loss of generality we may assume that $q_0 = 0$ and $h(0) = 0$. Following the proof of Proposition 3.5, we show that this implies

$$h(z, 0) \in \mathrm{span}_{\mathbb{C}}\{v_1(0, 0), \ldots, v_{n-k}(0, 0)\}. \tag{4.13}$$

Using the fact that $h(q) \in A$, (4.10) and (4.13), we observe that $h(z, 0)$ satisfies the following system of equations of rank $n - 1$, for z small enough,

$$\begin{cases} v_{j_n}(\bar{z}) = 2i\langle \tilde{v}_j(\bar{z}), \tilde{z}'\rangle, j = 1, \ldots, n-k, \\ \quad 0 \quad\ = \quad \langle a_s, \tilde{z}'\rangle, \quad s = 1, \ldots, k-1, \end{cases} \tag{4.14}$$

where $a_s \in \mathbb{C}^{n-1}, s = 1, \ldots, k-1$.

Using Cramer's rule again, we then solve (4.14) and obtain that each $h_j(z, 0), j = 1, \ldots, n$, is a quotient of two polynomials of degree at most $1 + \cdots + (n-k) = \dfrac{(n-k+1)(n-k)}{2}$. Therefore

$$\deg(h|_{Q_0}) \leq \frac{(n-k+1)(n-k)}{2}. \tag{4.15}$$

As in the first part of the proof, using Lemma 2.4 and Lemma 2.1, we obtain (4.1). The proof is now complete. □

Proof of Theorem 1.4. Theorem 1.4 is a direct consequence of Theorem 4.1. □

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